OPINION DYNAMICS: INHOMOGENEOUS BOLTZMANN-TYPE EQUATIONS MODELLING OPINION LEADERSHIP AND POLITICAL SEGREGATION

BERTRAM DÜRING[1] AND MARIE-THERESE WOLFRAM[2]

Abstract. We propose and investigate different kinetic models for opinion formation, when the opinion formation process depends on an additional independent variable, e.g. a leadership or a spatial variable. More specifically, we consider: (i) opinion dynamics under the effect of opinion leadership, where each individual is characterised not only by its opinion, but also by another independent variable which quantifies leadership qualities; (ii) opinion dynamics modelling political segregation in the 'The Big Sort', a phenomenon that US citizens increasingly prefer to live in neighbourhoods with politically like-minded individuals. Based on microscopic opinion consensus dynamics such models lead to inhomogeneous Boltzmann-type equations for the opinion distribution. We derive macroscopic Fokker-Planck-type equations in a quasi-invariant opinion limit and present results of numerical experiments.

1. Introduction

The dynamics of opinion formation have been studied with growing attention; in particular in the field of physics [15, 10, 24], in which a new research field termed *sociophysics* (going back to the pioneering work of Galam *et al.* [16]) emerged. More recently, different kinetic models to describe opinion formation have been proposed [25, 3, 12, 8, 6, 7, 20]. Such models successfully use methods from statistical mechanics to describe the behaviour of a large number of interacting individuals in a society. This leads to generalisations of the classical Boltzmann equation for gas dynamics. Then the framework from classical kinetic theory for homogeneous gases is adapted to the sociological setting by replacing molecules and their velocities by individuals and their opinion. Instead of binary collisions, one considers the process of compromise between two individuals.

The basic models typically assume a homogeneous society. To model additional sociologic effects in real societies, e.g. the influence of strong opinion leaders [12], one needs to consider *inhomogeneous* models. One solution, which arises naturally in certain situations (see, e.g. [11]), is to consider the time-evolution of distribution functions of different, interacting species. To some extent this can be seen as the analogue to the physical problem of a mixture of gases, where the molecules of the different gases exchange momentum during collisions [5]. This leads to systems of Boltzmann-like equations for the opinion distribution functions $f_i = f_i(w,t)$, $i = 1, \ldots, n$, of n interacting species which are of the form

$$\frac{\partial}{\partial t} f_i(w,t) = \sum_{j=1}^{n} \frac{1}{\tau_{ij}} \mathcal{Q}_{ij}(f_i, f_j)(w). \tag{1}$$

The Boltzmann-like collision operators $\mathcal{Q}_{ij}$ describe the change in time of $f_i(w,t)$ due to binary interaction depending on a balance between the gain and loss of individuals with opinion w. The suitably chosen relaxation times τ_{ij} allow to control the interaction frequencies. To model exchange of individuals (mass) between different species, additional collision operators can be present on the right hand side of (1), which are reminiscent of chemical reactions in the physical situation.

Another alternative is to study models where the distribution function depends on an additional variable as e.g. in [14]. This leads to *inhomogeneous Boltzmann-type equations* for the distribution

Key words and phrases. Boltzmann equation; Fokker-Planck equation; opinion formation; sociophysics.

function $f = f(x, w, t)$ which are of the following form

$$\frac{\partial}{\partial t} f + \mathrm{div}_x(\Phi(x, w) f) = \frac{1}{\tau} \mathcal{Q}(f, f). \tag{2}$$

Clearly, the choice of the field $\Phi = \Phi(x, w)$ which describes the opinion flux plays a crucial role. It may not be easy to determine a suitable field from the economic or sociologic problem, in contrast to the physical situation where the law of motion yields the right choice.

In this paper, we give two examples of opinion formation problems, which can be modelled using *inhomogeneous Boltzmann-type equations*. One is concerned with opinion formation where the compromise process depends on the interacting individuals' leadership abilities. The other considers the so-called 'Big Sort phenomena', the clustering of individuals with similar political opinion observed in the USA. Both problems lead to an inhomogeneous Boltzmann-type equation of the form (2).

Our work is based on a homogeneous kinetic model for opinion formation introduced by Toscani in [25]. The idea of this kinetic model is to describe the evolution of the distribution of opinion by means of *microscopic* interactions among individuals in a society. Opinion is represented as a continuous variable $w \in \mathcal{I}$ with $\mathcal{I} = [-1, 1]$, where ± 1 represent extreme opinions. If concerning political opinions $\mathcal{I}$ can be identified with the left-right political spectrum. Toscani bases his model on two main aspects of opinion formation. The first one is a *compromise process* [17, 10, 27], in which individuals tend to reach a compromise after exchange of opinions. The second one is *self-thinking,* where individuals change their opinion in a diffusive way, possibly influenced by exogenous information sources like the media. Based on both Toscani [25] defines a kinetic model in which opinion is exchanged between individuals through pairwise interactions: when two individuals with pre-interaction opinion v and w meet, then their post-trade opinions v^* and w^* are given by

$$v^* = v - \gamma P(|v - w|)(v - w) + \tilde{\eta} D(v), \quad w^* = w - \gamma P(|v - w|)(w - v) + \eta D(w). \tag{3}$$

Herein, $\gamma \in (0, 1/2)$ is the constant *compromise parameter*. The quantities $\tilde{\eta}$ and η are random variables with the same distribution with mean zero and variance σ^2. They model *self-thinking* which each individual performs in a random diffusion fashion through an exogenous, global access to information, e.g. through the press, television or internet. The functions $P(\cdot)$ and $D(\cdot)$ model the local relevance of compromise and self-thinking for a given opinion. To ensure that post-interaction opinions remain in the interval $\mathcal{I}$ additional assumptions need to be made on the random variables and the functions $P(\cdot)$ and $D(\cdot)$, see [25] for details. In this setting, the time-evolution of the distribution of opinion among individuals in a simple, homogeneous society is governed by a homogeneous Boltzmann-type equation of the form (1). In a suitable scaling limit, a partial differential equation of Fokker-Planck type is derived for the distribution of opinion. Similar diffusion equations are also obtained in [23] as a mean field limit the Sznajd model [24]. Mathematically, the model in [25] is related to works in the kinetic theory of granular gases [9]. In particular, the non-local nature of the compromise process is analogous to the variable coefficient of restitution in inelastic collisions [26]. Similar models are used in the modelling of wealth and income distributions which show Pareto tails, cf. [13] and the references therein.

The paper is organised as follows. We introduce two new, inhomogeneous models for opinion formation. In Section 2 we consider opinion formation dynamics which take into account the effect of opinion leadership. Each individual is not only characterised by its opinion but also by another independent variable, the assertiveness, which quantifies its leadership potential. Starting from a microscopic model for the opinion dynamics we arrive at an inhomogeneous Boltzmann-type equation for the opinion distribution function. We show that, alternatively similar dynamics can be modelled by a multi-dimensional kinetic opinion formation model. In Section 3 we turn to the modelling of 'The Big Sort', the phenomenon that US citizens increasingly prefer to live among others who share their political opinions. We propose a kinetic model of opinion formation which takes into account these preferences. Again, the time evolution of the opinion distribution is described by an inhomogeneous Boltzmann-type equation. In Section 4 we derive the corresponding macroscopic Fokker-Planck-type limit equations for the inhomogeneous Boltzmann-type equations

in a quasi-invariant opinion limit. Details on the numerical solvers as well as results of numerical experiments are presented in Section 5. Section 6 concludes.

2. Opinion formation and the influence of opinion leadership

The prevalent literature on opinion formation has focused on election processes, referendums or public opinion tendencies. With the exception of [3, 12] less attention has been paid to the important effect that opinion leaders have on the dissemination of new ideas and the diffusion of beliefs in a society. Opinion leadership is one of several sociological models trying to explain formation of opinions in a society. Certain, typical personal characteristics are supposed to characterise opinion leaders: high confidence, high self-esteem, a strong need to be unique, and the ability to withstand criticism. An opinion leader is socially active, highly connected and held in high esteem by those accepting his or her opinion. Opinion leaders appears in such different areas as political parties and movements, advertisement of commercial products and dissemination of new technologies.

In the opinion formation model in [12] the society is built of two groups, one group of opinion leaders, and one of ordinary people, so-called followers. In this model, individuals from the same group can influence each others opinions, but opinion leaders are assertive and, although able to influence followers, are unmoved by the followers' opinions. In this model a leader always remains a leader, a follower always a follower. Hence it is not possible to describe the emergence or deline of leaders.

In the model proposed in this section we assume that the leadership qualities (like assertiveness, self-confidence, ...) of each individual are characterised through an additional independent variable, to which we refer short as *assertiveness*. In the sociological literature the term assertiveness describes a person's tendency to actively defend, pursue and speak out for his or her own values preferences and goals. There has been a lot of research on how assertiveness is connected to leadership, see for example [2] and references therein.

2.1. An inhomogeneous Boltzmann-type equation.

Our approach is based on Toscani's model for opinion formation (3), but assumes that the compromise process is also influence by the assertiveness of the interacting individuals. The assertiveness of an individual is represented by the continuous variable $x \in \mathcal{J}$ with $\mathcal{J} = [-1, 1]$. The endpoints of the interval $\mathcal{J}$, i.e. ± 1, represent a strongly or weakly assertive individual. A leader would correspond to a strongly assertive individual.

The interaction for two individuals with opinion and assertiveness (v, x) and (w, y) reads as:

$$v^* = v - \gamma C(x, y, v, w)(v - w) + \tilde{\eta} D(v), \quad w^* = w - \gamma C(x, y, v, w)(w - v) + \eta D(w). \tag{4}$$

The first term on the right hand sides of (4) models the compromise process, the second the self-thinking process. The functions $C(\cdot)$ and $D(\cdot)$ model the local relevance of compromise and self-thinking for a given opinion, respectively. The constant *compromise parameter* $\gamma \in (0, 1/2)$ controls the 'speed' of attraction of two different opinions. The quantities $\tilde{\eta}$, η are random variables with distribution Θ with variance σ^2 and zero mean, assuming values on a set $\mathcal{B} \subset \mathbb{R}$.

Let us discuss the interaction described in (4) and its ingredients in more detail. In each such interaction, the pre-interaction opinion v increases (gets closer to w) when $v < w$ and decreases in the opposite situation; the change of the pre-interaction opinion w happens in a similar way. We assume that the compromise process function C can be written in the following form:

$$C(x, y, v, w) = R(x, x - y) P(|v - w|). \tag{5}$$

In (4) we will only allow interactions that guarantee $v^*, w^* \in \mathcal{I}$. To this end, we assume

$$0 \le P(|v - w|) \le 1, \quad 0 \le R(x, x - y) \le 1, \quad 0 \le D(v) \le 1.$$

Let us now discuss useful assumptions on the functions P, R and D.

As in [25] we assume that the ability to find a compromise is linked to the distance between opinions. The higher this distance is, the lower the possibility to find a compromise. Hence, the *localisation function* $P(\cdot)$ is assumed to be a decreasing function of its argument. Usual choices are $P(|v - w|) = \mathbf{1}_{\{|v-w| \le c\}}$ for some constant $c > 0$ and smoothed variants thereof [25].

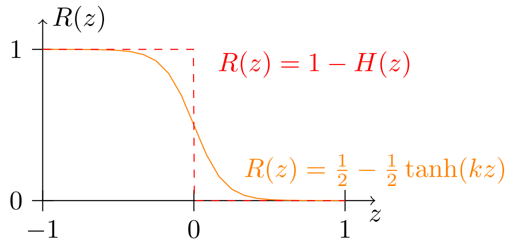


FIGURE 1. Choice of function R: in the limit $k \to \infty$ a step-function is approached, and the interaction effectively approaches a variant of the leader-follower model in [12] (if the assertiveness of individuals were assumed to be constant in time).

On the other hand, we assume the higher the assertiveness level, the lower the tendency of an individual to change their opinion. Hence, $R(\cdot)$ should be decreasing in its argument, too. A possible choice for $R(\cdot)$ may be

$$R(x, x-y) = R(x-y) = \frac{1}{2} - \frac{1}{2}\tanh\big(k(x-y)\big) \tag{6}$$

for some constant $k > 0$. This choice is motivated by the following considerations: Let A and B denote two individuals with assertiveness and opinion (x, v) and (y, w), respectively. Then the particular choice of (6) corresponds to the two cases:

- If $x \approx 1$ and $y \approx -1$, i.e. a highly assertive individual A meets a weakly assertive individual B: then $R(x-y) \approx 0$ (no influence of individual B on A), but $R(y-x) \approx 1$, i.e. the leader A persuades a weakly assertive individual B.
- If both individuals have a similar assertiveness level and hence $|x-y|$ is small, there will be some exchange of opinion, no matter how large (or small) this assertiveness level is.

Note that in the limit $k \to \infty$ the choice (6) corresponds to an approximation of $1 - H(z)$ with H denoting the Heaviside function, see Figure 1. In this limit, individuals are either maximally assertive in the interaction with a value of the assertiveness variable in $(0, 1)$, or minimally assertive with assertiveness in $(-1, 0)$. Effectively, one would recover in the same limit a variant of the opinion leader-follower model in [12] if the assertiveness of individuals were assumed to be constant in time.

We conclude by discussing the choice of $D(\cdot)$. We assume that the ability to change individual opinions by self-thinking decreases as one gets closer to the extremal opinions. This reflects the fact that extremal opinions are more difficult to change. Therefore, we assume that the self-thinking function $D(\cdot)$ is a decreasing function of v^2 with $D(1) = 0$. We also need to choose the set $\mathcal{B}$, i.e. we have to specify the range of values the random variables $\tilde{\eta}$, η in (4) can assume. Clearly, it depends on the particular choice for $D(\cdot)$. Let us consider the upper bound at $w = 1$ first. To ensure that individuals' opinions do not leave $\mathcal{I}$, we need

$$v^* = v - \gamma C(x, y, v, w)(v - w) + \tilde{\eta} D(v) \leq 1$$

Obviously, the worst case is $w = 1$, where we have to ensure

$$\tilde{\eta} D(v) \leq 1 - v + \gamma(v - 1) = (1 - v)(1 - \gamma)$$

Hence, if $D(v)/(1 - v) \leq K_+$ it suffices to have $|\tilde{\eta}| \leq (1 - \gamma)/K_+$. A similar computation for the lower boundary shows that if $D(v)/(1 + v) \leq K_-$ it suffices to have $|\tilde{\eta}| \leq (1 - \gamma)/K_-$.

In this setting, we are led to study the evolution of the distribution function as a function depending on the assertiveness $x \in \mathcal{J}$, opinion $w \in \mathcal{I}$ and time $t \in \mathbb{R}^+$, $f = f(x, w, t)$. In analogy with the classical kinetic theory of rarefied gases, we emphasise the role of the different independent variables by identifying the velocity with opinion, and the position with assertiveness. In this way, we assume at once that the variation of the distribution $f(x, w, t)$ with respect to the opinion variable w depends on 'collisions' between agents, while the time change of distributions

with respect to the assertiveness x depends on the transport term. Unlike in physical applications where the transport term involves the velocity field of particles, here the transport term contains an equivalent 'opinion-velocity field' $\Phi = \Phi(x, w)$ which controls the flux depending on the independent variables x and w. This is in contrast to the physical situation of rarefied gases where the field would simply be given by $\Phi(w) = w$.

The time-evolution of the distribution function $f = f(x, w, t)$ of individuals depending on assertiveness $x \in \mathcal{J}$, opinion $w \in \mathcal{I}$ and time $t \in \mathbb{R}^+$ will obey an *inhomogeneous Boltzmann-type equation*,

$$\frac{\partial}{\partial t} f(x, w, t) + \operatorname{div}_x(\Phi(x, w) f(x, w, t)) = \frac{1}{\tau} \mathcal{Q}(f, f)(x, w, t). \tag{7}$$

Herein, Φ is the 'opinion-velocity field' and τ is a suitable relaxation time which allows to control the interaction frequency. The Boltzmann-like collision operator $\mathcal{Q}$ which describes the change of density due to binary interactions is derived by standard methods of kinetic theory. A useful way of writing the collision operators is the so-called weak form. It corresponds to consider, for all smooth functions $\phi(w)$,

$$\begin{aligned} &\int_{\mathcal{J}} \int_{\mathcal{I}} \mathcal{Q}(f, f)(x, w, t) \phi(w)\, dw\, dx \\ =&\frac{1}{2} \left\langle \int_{\mathcal{J}} \int_{\mathcal{I}^2} \big(\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)\big) f(x, v, t) f(y, w, t)\, dv\, dw\, dx \right\rangle, \end{aligned} \tag{8}$$

where $\langle \cdot \rangle$ denotes the operation of mean with respect to the random quantities $\tilde{\eta}$, η.

Choosing $\phi(w) = 1$ as test function in (8) and denoting the mass by

$$\rho(f)(t) = \int_{\mathcal{J}} \int_{\mathcal{I}} f(x, w, t)\, dw\, dx$$

implies conservation of mass,

$$\frac{\mathrm{d}\rho(f)(t)}{\mathrm{d}t} = \frac{1}{\tau} \int_{\mathcal{J}} \int_{\mathcal{I}} \mathcal{Q}(f, f)(x, w, t)\, dw\, dx = 0.$$

If there is point-wise conservation of opinion in each interaction in (4) (e.g. choosing $C \equiv 1$ and $D \equiv 0$),

$$v^* + w^* = v + w$$

or, more generally, conservation of opinion in the mean in each interaction in (4) (e.g. choosing $C \equiv 1$),

$$\langle v^* + w^* \rangle = v + w,$$

then the first moment, the mean opinion, is also conserved. This can be seen by choosing $\phi(w) = w$ in (8). Denoting the mean opinion by

$$m(f)(t) = \int_{\mathcal{J}} \int_{\mathcal{I}} f(x, w, t) w\, dw\, dx,$$

we obtain

$$\frac{\mathrm{d}m(f)(t)}{\mathrm{d}t} = \frac{1}{\tau} \int_{\mathcal{J}} \int_{\mathcal{I}} \mathcal{Q}(f, f)(x, w, t) w\, dw\, dx = 0.$$

We still have to specify the 'opinion-velocity field' $\Phi(x, w)$. A possible choice can be

$$\Phi(x, w) = G(x, w, t)(1 - x^2)^{\alpha}, \tag{9}$$

where $G = G(x, w, t)$ models the in- or decrease of the assertiveness level, while the prefactor $(1 - x^2)^{\alpha}$ ensures that the assertiveness level stays inside the domain $\mathcal{J}$.

This leads to an *inhomogeneous Boltzmann-type equation* of the following form

$$\frac{\partial}{\partial t} f + \operatorname{div}_x(G(x, w, t)(1 - x^2)^{\alpha} f) = \frac{1}{\tau} \mathcal{Q}(f, f). \tag{10}$$

2.2. **A multidimensional Boltzmann-type equation.** Alternatively, we can consider that both, change of opinion and and change of the assertiveness level happen through binary collisions. In this case we have the following interaction rules:

$$v^* = v - \gamma R(x, x-y) P(|v-w|)(v-w) + \tilde{\eta} D(v), \tag{11a}$$

$$x^* = x + \delta \tilde{G}(x, x-y) P(|v-w|)(x-y) + \mu D(x), \tag{11b}$$

$$w^* = w - \gamma R(y, y-x) P(|v-w|)(w-v) + \eta D(w), \tag{11c}$$

$$y^* = y + \delta \tilde{G}(y, y-x) P(|v-w|)(y-x) + \tilde{\mu} D(y). \tag{11d}$$

Herein, the functions P, R, and D play the same role as in the previous section, and are assumed to fulfil the assumptions introduced earlier. The constant parameters $\gamma \in (0, 1/2)$ and $\delta \in (0, 1/2)$ control the 'speed' of attraction of opinions and repulsion of assertiveness, respectively. In addition, we introduce the function $\tilde{G}$ which describes the in- and decrease of the individual assertiveness level. It is assumed to fulfil

$$0 \leq \tilde{G}(x, x-y) \leq 1.$$

Together, these assumptions guarantee that $v^*, w^* \in \mathcal{I}$ and $y^*, x^* \in \mathcal{J}$. The quantities $\tilde{\eta}$, η and $\tilde{\mu}$, μ are uncorrelated random variables with distribution Θ with variances σ_η^2 and σ_μ^2, respectively, and zero mean, assuming values on a set $\mathcal{B} \subset \mathbb{R}$.

To specify $\tilde{G}$, we make the assumption that highly assertive individuals gain more assertiveness and weakly assertive ones loose assertiveness in a collision. Therefore, we propose the following form:

$$\tilde{G}(x, x-y) = (1-x^2)^\alpha |x-y|, \tag{12}$$

where the quadratic polynomial ensures that the assertiveness level remains in $\mathcal{J}$.

In this setting, we are led to study the evolution of the distribution function as a function depending on the assertiveness $x \in \mathcal{J}$, opinion $w \in \mathcal{I}$ and time $t \in \mathbb{R}^+$, $f = f(x, w, t)$. Different to the previous section, we assume that the variation of the distribution $f(x, w, t)$ with respect to both variables, assertiveness x and opinion w, depends on collisions between individuals. The time-evolution of the distribution function $f = f(x, w, t)$ of individuals depending on assertiveness $x \in \mathcal{J}$, opinion $w \in \mathcal{I}$ and time $t \in \mathbb{R}^+$ will obey a *multi-dimensional homogeneous Boltzmann-type equation*,

$$\frac{\partial}{\partial t} f(x, w, t) = \frac{1}{\tau} \mathcal{Q}(f, f)(x, w, t), \tag{13}$$

where τ is a suitable relaxation time which allows to control the interaction frequency. The Boltzmann-like collision operator $\mathcal{Q}$ which describes the change of density due to binary interactions is derived by standard methods of kinetic theory. The weak form of (13) is given by

$$\begin{aligned}
&\int_{\mathcal{J}} \int_{\mathcal{I}} \mathcal{Q}(f, f)(x, w, t) \phi(x, w) \, dw \, dx \\
&\quad = \frac{1}{2} \left\langle \int_{\mathcal{J}^2} \int_{\mathcal{I}^2} \big(\phi(x^*, w^*) + \phi(y^*, v^*) - \phi(x, w) - \phi(y, v) \big) f(x, v, t) f(y, w, t) \, dv \, dw \, dy \, dx \right\rangle,
\end{aligned}$$

for all test functions $\phi(x, w)$ where $\langle \cdot \rangle$ denotes the operation of mean with respect to the random quantities $\tilde{\eta}$, η and $\tilde{\mu}$, μ.

3. Political segregation: 'The Big Sort'

In this section we are interested in modelling the clustering of individuals who share similar political opinions, a process that has been observed in the USA over the last decades.

In 2008, journalist Bill Bishop achieved popularity as the author of the book *The Big Sort: Why the Clustering of Like-Minded America Is Tearing Us Apart* [4]. Bishop's thesis is that US citizens increasingly choose to live among politically like-minded neighbours. Based on county-level election results of US presidential elections in the past thirty years, he observed a doubling of so-called 'landslide counties', counties in which either candidate won or lost by 20 percentage

points or more. Such a segregation of political supporters may result in making political debates more bitter and hamper the political decision-making by consensus.

Bishop's findings were discussed and acclaimed in many newspapers and magazines, and former president Bill Clinton urged audiences to read the book. On the other hand his claims also met opposition from political sociologists [1] who argued that political segregation is only a by-product of the correlation of political opinions with other sociologic (cultural background, race, ...) and economic factors which drive citizens residential preferences. One could consider to include such additional factors in a generalised kinetic model, and potentially even couple the opinion dynamics with a kinetic model for wealth distribution [14, 13].

'The Big Sort' lends itself as an ideal subject to study inhomogeneous kinetic models for opinion formation. In this context we are faced with spatial inhomogeneity rather than inhomogeneity in assertiveness as in Section 2. In the following we propose a kinetic model for opinion formation when the individuals are driven towards others with a similar political opinion.

3.1. **An inhomogeneous Boltzmann-type equation.** We study the evolution of the distribution function of political opinion as a function depending on three independent variables, the continuous political opinion variable $w \in [-1, 1]$, the position $x \in \mathbb{R}^2$ on our (virtual) map and time $t \in \mathbb{R}^+$, $f = f(x, w, t)$. In analogy with the classical kinetic theory of rarefied gases, we assume that the variation of the distribution $f(x, w, t)$ with respect to the political opinion variable w depends on collisions between individuals, while the time change of distributions with respect to the position x depends on the transport term, which contains the 'opinion-velocity field' $\Phi = \Phi(x, w)$. The specific choice of Φ is discussed further below.

The exchange of opinion is modelled by binary collisions in the operator $\mathcal{Q}$ and has a similar structure as (11). Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote the positions of two individuals with opinion v and w, respectively. Then the interaction rule reads as:

$$v^* = v - \gamma K(|x - y|)P(|v - w|)(v - w) + \tilde{\eta} D(v), \tag{14a}$$

$$w^* = w - \gamma K(|y - x|)P(|v - w|)(w - v) + \eta D(w). \tag{14b}$$

In (14) we make the following assumptions:

- Individuals exchange opinion if they are close to each other, i.e. their physical distance is less than a certain radius. This is modelled by the function $K(z) = \mathbf{1}_{\{|z| \leq r\}}$ with some radius $r > 0$.
- The functions P and D and the random quantities $\tilde{\eta}$, η satisfy the same assumptions as in Section 2.

We still need to specify the 'opinion-velocity field' $\Phi(x, w)$. As as first approach it helps to think of $\Phi(x, w)$ modelling the movement of individuals with respect to a given initial configuration: we consider $\mathbb{R}^2$ and a number of counties Ω_i which constitute a disjoint cover of $\mathbb{R}^2$, each with a county seat $c_i = (c_i^{(1)}, c_i^{(2)}) \in \mathbb{R}^2 \times [-1, 1]$ with position in $\mathbb{R}^2$ and an election result (by plurality voting system) which is either 'red/Republican' ($w > 0$) or 'blue/Democratic' ($w < 0$). In our numerical experiments presented later we will typically restrict ourselves to bounded domains $\Omega \subset \mathbb{R}^2$ with no-flow boundary conditions.

We assume that supporters of a party are attracted to counties which are controlled by the party they support. We model this effect by defining $\Phi(x, w)$ as a potential that drives the dynamics and is given by a superposition of (signed) Gaussians $C(x)$ centered around $c_i^{(1)}$ with variance σ_i. We assume also that stronger supporters, i.e. individuals with more extreme opinion values, are able to retain there positions. A possible way to take these effects into account is to choose

$$\Phi(x, w) = \operatorname{sign}(w) \nabla C(x)(1 - |x|^\beta). \tag{15}$$

The time-evolution of the distribution function $f = f(x, w, t)$ of individuals with political opinion $w \in \mathcal{I}$ at position $x \in \mathbb{R}^2$ at time $t \in \mathbb{R}^+$ will obey an *inhomogeneous Boltzmann-type equation* for the distribution function $f = f(x, w, t)$ which is of the form

$$\frac{\partial}{\partial t} f + \operatorname{div}_x(\Phi(x, w) f) = \frac{1}{\tau} \mathcal{Q}(f, f). \tag{16}$$

In a two-party system as we consider it, quantities of interest are the supporters of the parties, given by the marginals

$$f_D(x,t) = \int_{-1}^{0} f(x,w,t)\, dw, \quad f_R(x,t) = \int_{0}^{1} f(x,w,t)\, dw. \tag{17}$$

In simulations, elections could be run at time t by computing

$$D_i(t) = \int_{\Omega_i} f_D(x,t)\, dx = \int_{\Omega_i}\int_{-1}^{0} f(x,w,t)\, dw\, dx, \tag{18}$$

$$R_i(t) = \int_{\Omega_i} f_R(x,t)\, dx = \int_{\Omega_i}\int_{0}^{1} f(x,w,t)\, dw\, dx, \tag{19}$$

to adapt the values of $c_i^{(2)}$ defining the attractive Gaussians $C(x)$ in (15) accordingly in time.

4. Fokker-Planck limits

4.1. **Fokker-Planck limit for the inhomogeneous Boltzmann equation.** In general, equations like (16) (and (10)) are rather difficult to treat and it is usual in kinetic theory to study certain asymptotics, which frequently lead to simplified models of Fokker-Planck type. To this end, we study by formal asymptotics the quasi-invariant opinion limit ($\gamma, \sigma_\eta \to 0$ while keeping $\sigma_\eta^2/\gamma = \lambda$ fixed), following the path laid out in [25].

Let us introduce some notation. First, consider test-functions $\phi \in \mathcal{C}^{2,\delta}([-1,1])$ for some $\delta > 0$. We use the usual Hölder norms

$$\|\phi\|_\delta = \sum_{|\alpha|\le 2} \|D^\alpha \phi\|_{\mathcal{C}} + \sum_{\alpha=2} [D^\alpha \phi]_{\mathcal{C}^{0,\delta}},$$

where $[h]_{\mathcal{C}^{0,\delta}} = \sup_{v\neq w} |h(v)-h(w)|/|v-w|^\delta$. Denoting by $\mathcal{M}_0(A)$, $A \subset \mathbb{R}$ the space of probability measures on A, we define

$$\mathcal{M}_p(A) = \left\{ \Theta \in \mathcal{M}_0 \;\middle|\; \int_A |\eta|^p d\Theta(\eta) < \infty,\, p \ge 0 \right\},$$

the space of measures with finite pth momentum. In the following all our probability densities belong to $\mathcal{M}_{2+\delta}$ and we assume that the density Θ is obtained from a random variable Y with zero mean and unit variance. We then obtain

$$\int_{\mathcal{I}} |\eta|^p \Theta(\eta)\, d\eta = \mathrm{E}[|\sigma_\eta Y|^p] = \sigma_\eta^p \mathrm{E}[|Y|^p], \tag{20}$$

where $\mathrm{E}[|Y|^p]$ is finite. The weak form of (16) is given by

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{I}\times\mathbb{R}^2} f(x,w,t)\phi(w)\, dw\, dx + \int_{\mathcal{I}\times\mathbb{R}^2} \mathrm{div}_x(\Phi(x,w) f(x,w,t))\phi(w)\, dw\, dx \\ = \frac{1}{\tau}\int_{\mathcal{I}\times\mathbb{R}^2} \mathcal{Q}(f,f)(w)\phi(w)\, dw\, dx \end{aligned} \tag{21}$$

where

$$\begin{aligned} &\int_{\mathcal{I}\times\mathbb{R}^2} \mathcal{Q}(f,f)(w)\phi(w)\, dw\, dx \\ &\quad = \frac{1}{2}\left\langle \int_{\mathcal{I}^2\times\mathbb{R}^2} \big(\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)\big) f(x,v) f(x,w)\, dv\, dw\, dx \right\rangle. \end{aligned}$$

To study the situation for large times, i.e. close to the steady state, we introduce for $\gamma \ll 1$ the transformation

$$\tilde{t} = \gamma t,\; \tilde{x} = \gamma x,\; g(\tilde{x}, w, \tilde{t}) = f(x,w,t).$$

This implies $f(x,w,0) = g(\tilde{x},w,0)$ and the evolution of the scaled density $g(\tilde{x},w,\tilde{t})$ follows (we immediately drop the tilde in the following and denote the rescaled variables simply by t and x)

$$\begin{aligned}(22)\quad \frac{d}{dt}\int_{\mathcal{I}\times\mathbb{R}^2} g(x,w,t)\phi(w)\,dw\,dx + \int_{\mathcal{I}\times\mathbb{R}^2} \mathrm{div}_x(\Phi(x,w)g(x,w,t))\phi(w)\,dw\,dx \\ = \frac{1}{\gamma\tau}\int_{\mathcal{I}\times\mathbb{R}^2} \mathcal{Q}(g,g)(w)\phi(w)\,dw\,dx.\end{aligned}$$

Due to the collision rule (4), it holds

$$w^* - w = -\gamma C(x,y,v,w)(w-v) + \eta D(w) \ll 1.$$

Taylor expansion of ϕ up to second order around w of the right hand side of (22) leads to

$$\begin{aligned}&\Big\langle \frac{1}{\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi'(w)\left[-\gamma C(x,y,v,w)(w-v) + \eta D(w)\right] g(x,w)g(x,v)\,dv\,dw\,dx\Big\rangle \\ &\quad + \Big\langle \frac{1}{2\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi''(\tilde{w})\left[-\gamma C(x,y,v,w)(w-v) + \eta D(w)\right]^2 g(x,w)g(x,v)\,dv\,dw\,dx\Big\rangle \\ &= \frac{1}{\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi'(w)\left[-\gamma C(x,y,v,w)(w-v))\right] g(x,w)g(x,v)\,dv\,dw\,dx \\ &\quad + \Big\langle \frac{1}{2\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi''(w)\big[\gamma C(x,y,v,w)(w-v) + \eta D(w)\big]^2 g(x,w)g(x,v)\,dv\,dw\,dx\Big\rangle + R(\gamma,\sigma_\eta) \\ &= -\frac{1}{\tau}\int_{\mathcal{I}\times\mathbb{R}^2} \phi'(w)\mathcal{K}(x,w)g(x,w)\,dw\,dx \\ &\quad + \frac{1}{2\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi''(w)\Big[\gamma^2 C^2(x,y,v,w)(w-v)^2 + \gamma\lambda D^2(w)\Big] g(x,w)g(x,v)\,dv\,dw\,dx + R(\gamma,\sigma_\eta),\end{aligned}$$

with $\tilde{w} = \kappa w^* + (1-\kappa)w$ for some $\kappa \in [0,1]$ and

$$\begin{aligned}R(\gamma,\sigma_\eta) = \Big\langle \frac{1}{2\gamma\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} &(\phi''(\tilde{w}) - \phi''(w)) \\ &\times \left[-\gamma C(x,y,v,w)(w-v) + \eta D(w)\right]^2 g(x,w)g(x,v)\,dv\,dw\,dx\Big\rangle\end{aligned}$$

and

$$\mathcal{K}(x,w) = \int_{\mathcal{I}} C(x,y,v,w)(w-v)g(x,v)\,dv.$$

Now we consider the limit $\gamma,\sigma_\eta \to 0$ while keeping $\lambda = \sigma_\eta^2/\gamma$ fixed.

We first show that the remainder term $R(\gamma,\sigma_\eta)$ vanishes is this limit, similar as in [25]. Note first that as $\phi \in \mathcal{F}_{2+\delta}$, by the collision rule (4) and the definition of $\tilde{w}$ we have

$$|\phi''(\tilde{w}) - \phi''(w)| \le \|\phi''\|_\delta|\tilde{w} - w|^\delta \le \|\phi''\|_\delta|w^* - w|^\delta = \|\phi''\|_\delta\left|\gamma C(x,y,v,w)(w-v) + \eta D(w)\right|^\delta.$$

Thus we obtain

$$R(\gamma,\sigma_\eta) \le \frac{\|\phi''\|_\delta}{2\gamma\tau}\Big\langle \int_{\mathcal{I}^2\times\mathbb{R}^2} \left[-\gamma C(x,y,v,w)(w-v) + \eta D(w)\right]^{2+\delta} g(x,w)g(x,v)\,dv\,dw\,dx\Big\rangle.$$

Furthermore, we note that

$$\begin{aligned}&[\eta D(w) - \gamma C(x,y,v,w)(w-v)]^{2+\delta} \le 2^{1+\delta}\left(|\gamma C(x,y,v,w)(w-v)|^{2+\delta} + |\eta D(w)|^{2+\delta}\right) \\ \le &2^{3+2\delta}|\gamma|^{2+\delta} + 2^{1+\delta}|\eta|^{2+\delta}.\end{aligned}$$

Here, we used the convexity of $f(s) := |s|^{2+\delta}$ and the fact that $w,v \in \mathcal{I}$ and thus bounded. We conclude

$$|R(\gamma,\sigma_\eta)| \le \frac{C\|\phi''\|_\delta}{\tau}\left(\gamma^{1+\delta} + \frac{1}{2\gamma}\left\langle|\eta|^{2+\delta}\right\rangle\right) = \frac{C\|\phi''\|_\delta}{\tau}\left(\gamma^{1+\delta} + \frac{1}{2\gamma}\int_{\mathcal{I}}|\eta|^{2+\delta}\Theta(\eta)\,d\eta\right).$$

Since $\Theta \in \mathcal{M}_{2+\delta}$, and η has variance σ_η^2 we have (see (20))

$$\int_{\mathcal{I}} |\eta|^{2+\delta}\Theta(\eta)\, d\eta = \mathrm{E}\left[\left|\sqrt{\lambda\gamma}Y\right|^{2+\delta}\right] = (\lambda\gamma)^{1+\frac{\delta}{2}}\mathrm{E}\left[|Y|^{2+\delta}\right]. \tag{23}$$

Thus, we conclude that the term $R(\gamma, \sigma_\eta)$ vanishes in the limit $\gamma, \sigma_\eta \to 0$ while keeping $\lambda = \sigma_\eta^2/\gamma$ fixed.

Then, in the same limit, the term on the right hand side of (22) converges to

$$\begin{aligned}
-\frac{1}{\tau}\int_{\mathcal{I}\times\mathbb{R}^2} &\phi'(w)\mathcal{K}(x,w)g(x,w)\,dw\,dx + \frac{1}{2\tau}\int_{\mathcal{I}^2\times\mathbb{R}^2} \phi''(w)\left[\lambda D^2(w)\right] g(x,w)g(x,v)\,dv\,dw\,dx \\
&= -\frac{1}{\tau}\int_{\mathcal{I}\times\mathbb{R}^2} \phi'(w)\mathcal{K}(x,w)g(x,w)\,dw\,dx + \frac{\lambda}{2\tau}\int_{\mathcal{I}\times\mathbb{R}^2} \phi''(w)M(x)D^2(w)g(x,w)\,dw\,dx,
\end{aligned}$$

with $M(x) = \int g(x,v)\,dv$ being the mass of individuals in x. After integration by parts we obtain the right hand side of (the weak form of) the Fokker-Planck equation

$$\begin{aligned}
&\frac{\partial}{\partial\tau}g(x,w,t) + \mathrm{div}_x(\Phi(x,w)g(x,w,t)) \\
&\qquad = \frac{\partial}{\partial w}\left(\frac{1}{\tau}\mathcal{K}(x,w,t)g(x,w,t)\right) + \frac{\lambda M(x)}{2\tau}\frac{\partial^2}{\partial w^2}\left(D^2(w)g(x,w,t)\right),
\end{aligned} \tag{24}$$

subject to no flux boundary conditions for the variable w (which result from the integration by parts).

4.2. **Fokker-Planck limit for the multidimensional model.** We follow a similar approach as in the previous section, now in the two-dimensional setting of the opinion formation model (11) (cf. also [22]). Our aim is to study by formal asymptotics the quasi-invariant opinion limit where $\gamma, \delta, \sigma_\eta, \sigma_\mu \to 0$ while keeping $\sigma_\eta^2/\gamma = c_1\sigma_\mu^2/(c_2\delta) = \lambda$ fixed, introducing the positive constants $c_1 = \delta/\gamma$ and $c_2 = \sigma_\mu/\sigma_\eta$.

We consider test-functions $\phi \in \mathcal{C}^{2,\delta}([-1,1]^2)$ for some $\delta > 0$. As above we use the usual Hölder norms and consider denote by $\mathcal{M}_p(A)$, $A \subset \mathbb{R}$ the space of probability measures on A with finite pth momentum. In the following all our probability densities belong to $\mathcal{M}_{2+\delta}$ and we assume that the density Θ is obtained from a random variable Y with zero mean and unit variance. We then obtain

$$\int_{\mathcal{I}} |\eta|^p\Theta(\eta)\; d\eta = \mathrm{E}[|\sigma Y|^p] = \sigma_\eta^p\mathrm{E}[|Y|^p], \quad \int_{\mathcal{I}} |\mu|^p\Theta(\mu)\; d\mu = \mathrm{E}[|\tilde\sigma Y|^p] = \sigma_\mu^p\mathrm{E}[|Y|^p],$$

where $\mathrm{E}[|Y|^p]$ is finite. The weak form of (13) is given by

$$\frac{d}{dt}\int_{\mathcal{J}\times\mathcal{I}} f(x,w,t)\phi(x,w)\,dw\,dx = \frac{1}{\tau}\int_{\mathcal{J}\times\mathcal{I}} \mathcal{Q}(f,f)(x,w)\phi(x,w)\,dw\,dx \tag{25}$$

where

$$\begin{aligned}
&\int_{\mathcal{J}\times\mathcal{I}} \mathcal{Q}(f,f)(x,w)\phi(x,w)\,dw\,dx \\
&\qquad = \frac{1}{2}\left\langle\int_{\mathcal{J}^2\times\mathcal{I}^2} \big(\phi(x^*,w^*) + \phi(y^*,v^*) - \phi(x,w) - \phi(y,v)\big) f(x,v)f(x,w)\,dv\,dy\,dw\,dx\right\rangle.
\end{aligned}$$

To study the situation for large times, i.e. close to the steady state, we introduce for $\gamma \ll 1$ the transformation

$$\tilde t = \gamma t,\;\; g(x,w,\tilde t) = f(x,w,t).$$

This implies $f(x,w,0) = g(x,w,0)$ and the evolution of the scaled density $g(x,w,\tilde t)$ follows (we immediately drop the tilde in the following and denote the rescaled time simply by t)

$$\frac{d}{dt}\int_{\mathcal{J}\times\mathcal{I}} g(x,w,t)\phi(x,w)\,dw\,dx = \frac{1}{\gamma\tau}\int_{\mathcal{J}\times\mathcal{I}} \mathcal{Q}(g,g)(x,w)\phi(x,w)\,dw\,dx. \tag{26}$$

Due to the collision rules (11), it holds

$$w^* - w = -\gamma R(x-y)P(|w-v|)(w-v) + \eta D(w) \ll 1,$$
$$x^* - x = \delta \tilde{G}(x-y)P(|v-w|)(x-y) + \mu D(x) \ll 1.$$

We now employ multidimensional Taylor expansion of ϕ up to second order around (x, w) of the right hand side of (26). Recalling that the random quantities η, μ have mean zero, variance σ_η and σ_μ, respectively, and are uncorrelated, we can follow along the lines of the computations of the previous subsection, to obtain

$$\begin{aligned}
&\Big\langle \frac{1}{\gamma\tau} \int_{\mathcal{J}^2 \times \mathcal{I}^2} \frac{\partial \phi}{\partial w}(x,w)\, [-\gamma R(x-y)P(|w-v|)(w-v) + \eta D(w)]\, g(x,w)g(x,v)\, dv\, dy\, dw\, dx \Big\rangle \\
&\qquad + \Big\langle \frac{1}{\gamma\tau} \int_{\mathcal{J}^2 \times \mathcal{I}^2} \frac{\partial \phi}{\partial x}(x,w)\, \Big[\delta \tilde{G}(x-y)P(|v-w|)(x-y) + \mu D(x)\Big] \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \Big\rangle \\
&\qquad + \Big\langle \frac{1}{2\gamma\tau} \int_{\mathcal{J}^2 \times \mathcal{I}^2} \frac{\partial^2 \phi}{\partial w^2}(x,w)\, [-\gamma R(x-y)P(|w-v|)(w-v) + \eta D(w)]^2 \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \Big\rangle \\
&\qquad + \Big\langle \frac{1}{2\gamma\tau} \int_{\mathcal{J}^2 \times \mathcal{I}^2} \frac{\partial^2 \phi}{\partial x^2}(x,w)\, \Big[\delta \tilde{G}(x-y)P(|v-w|)(x-y) + \mu D(x)\Big]^2 \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \Big\rangle \\
&\qquad + \Big\langle \frac{1}{2\gamma\tau} \int_{\mathcal{J}^2 \times \mathcal{I}^2} \frac{\partial^2 \phi}{\partial w \partial x}(x,w)\, [-\gamma R(x-y)P(|w-v|)(w-v) + \eta D(w)] \\
&\qquad\qquad \times \Big[\delta \tilde{G}(x-y)P(|v-w|)(x-y) + \mu D(x)\Big]\, g(x,w)g(x,v)\, dv\, dy\, dw\, dx \Big\rangle \\
&\qquad\qquad + R(\gamma, \sigma_\eta, \sigma_\mu) \\
&= -\frac{1}{\tau} \int_{\mathcal{I} \times \mathcal{J}^2} \frac{\partial \phi}{\partial w}(x,w) \mathcal{K}(x,w) g(x,w)\, dy\, dw\, dx - \frac{\delta}{\gamma\tau} \int_{\mathcal{I}^2 \times \mathcal{J}} \frac{\partial \phi}{\partial x}(x,w) \mathcal{L}(x,w) g(x,w)\, dv\, dw\, dx \\
&\qquad + \frac{1}{2\gamma\tau} \int_{\mathcal{I}^2 \times \mathcal{J}^2} \frac{\partial^2 \phi}{\partial w^2}(x,w) \Big[\gamma^2 R^2(x-y) P^2(|w-v|)(w-v)^2 + \sigma_\eta^2 D^2(w)\Big] \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \\
&\qquad + \frac{1}{2\gamma\tau} \int_{\mathcal{I}^2 \times \mathcal{J}^2} \frac{\partial^2 \phi}{\partial x^2}(x,w) \Big[\delta^2 \tilde{G}^2(x-y) P^2(|v-w|)(x-y)^2 + \sigma_\mu^2 D^2(x)\Big] \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \\
&\qquad + \frac{1}{2\gamma\tau} \int_{\mathcal{I}^2 \times \mathcal{J}^2} \frac{\partial^2 \phi}{\partial w \partial x}(x,w) \Big[\gamma\delta R(x-y) P^2(|w-v|)(w-v) \tilde{G}(x-y)(x-y)\Big] \\
&\qquad\qquad \times g(x,w)g(x,v)\, dv\, dy\, dw\, dx \\
&\qquad\qquad + R(\gamma, \sigma_\eta, \sigma_\mu),
\end{aligned}$$

where

$$\mathcal{K}(x,w) = \int_{\mathcal{I}} R(x-y)P(|w-v|)(w-v)g(x,v)\, dv, \tag{27a}$$

$$\mathcal{L}(x,w) = \int_{\mathcal{J}} \tilde{G}(x-y)P(|v-w|)(x-y)g(x,v)\, dy. \tag{27b}$$

and $R(\gamma, \sigma_\eta, \sigma_\mu)$ is the remainder term. Our aim is now to consider the formal limit $\gamma, \delta, \sigma_\eta, \sigma_\mu \to 0$ while keeping $\sigma_\eta^2/\gamma = c_1\sigma_\mu^2/(c_2\delta) = \lambda$ fixed, recalling $c_1 = \delta/\gamma$ and $c_2 = \sigma_\mu/\sigma_\eta$. The remainder term $R(\gamma, \sigma_\eta, \sigma_\mu)$ depends on the higher moments of the (uncorrelated) random quantities and

can be shown to vanish in this limit, similar as in the previous subsection (we omit the details). In the same limit, the term on the right hand side of (26) then converges to

$$
\begin{aligned}
&-\frac{1}{\tau}\int_{\mathcal{I}\times\mathcal{J}^2}\frac{\partial\phi}{\partial w}(x,w)\mathcal{K}(x,w)g(x,w)\,dy\,dw\,dx-\frac{c_1}{\tau}\int_{\mathcal{I}^2\times\mathcal{J}}\frac{\partial\phi}{\partial x}(x,w)\mathcal{L}(x,w)g(x,w)\,dv\,dw\,dx\\
&\qquad+\frac{1}{2\tau}\int_{\mathcal{I}^2\times\mathcal{J}^2}\frac{\partial^2\phi}{\partial w^2}(x,w)\lambda D^2(w)g(x,w)g(x,v)\,dv\,dy\,dw\,dx\\
&\qquad\qquad+\frac{1}{2\tau}\int_{\mathcal{I}^2\times\mathcal{J}^2}\frac{\partial^2\phi}{\partial x^2}(x,w)c_2\lambda D^2(x)g(x,w)g(x,v)\,dv\,dy\,dw\,dx\\
&=-\frac{1}{\tau}\int_{\mathcal{I}\times\mathcal{J}^2}\frac{\partial\phi}{\partial w}(x,w)\mathcal{K}(x,w)g(x,w)\,dy\,dw\,dx-\frac{c_1}{\tau}\int_{\mathcal{I}^2\times\mathcal{J}}\frac{\partial\phi}{\partial x}(x,w)\mathcal{L}(x,w)g(x,w)\,dv\,dw\,dx\\
&\qquad+\frac{\lambda}{2\tau}\int_{\mathcal{I}\times\mathcal{J}^2}\frac{\partial^2\phi}{\partial w^2}(x,w)M(x)D^2(w)g(x,w)\,dy\,dw\,dx\\
&\qquad\qquad+\frac{c_2\lambda}{2\tau}\int_{\mathcal{I}\times\mathcal{J}^2}\frac{\partial^2\phi}{\partial x^2}(x,w)M(x)D^2(x)g(x,w)\,dy\,dw\,dx,
\end{aligned}
$$

with $M(x)=\int g(x,v)\,dv$ being the mass of individuals in x. After integration by parts we obtain the right hand side of (the weak form of) the Fokker-Planck equation

$$
\begin{aligned}
\frac{\partial}{\partial\tau}g(x,w,t)&=\frac{1}{\tau}\frac{\partial}{\partial w}\left(\mathcal{K}(x,w,t)g(x,w,t)\right)+\frac{c_1}{\tau}\frac{\partial}{\partial x}\left(\mathcal{L}(x,w,t)g(x,w,t)\right)\\
&\quad+\frac{\lambda M(x)}{2\tau}\frac{\partial^2}{\partial w^2}\left(D^2(w)g(x,w,t)\right)+\frac{c_2\lambda M(x)}{2\tau}\frac{\partial^2}{\partial x^2}\left(D^2(x)g(x,w,t)\right),
\end{aligned}
\tag{28}
$$

subject to no flux boundary conditions which result from the integration by parts, with the nonlocal operators $\mathcal{K}$, $\mathcal{L}$ given in (27).

5. Numerical experiments

In this section, we illustrate the behaviour of the different kinetic models and the limiting Fokker-Planck-type equations with various simulations. We first discuss the numerical discretisation of the different Boltzmann-type equations and the corresponding Fokker-Planck-type equations, and then present results of numerical experiments.

While the multi-dimensional Boltzmann-type equation (13) can be solved by a classical kinetic Monte Carlo method, the Monte Carlo simulations for the inhomogeneous Boltzmann-type equations (2) are more involved. On the macroscopic level the high-dimensionality poses a significant challenge – hence we propose a time splitting as well as a finite element discretisation with mass lumping in space.

5.1. **Monte Carlo simulations for the multi-dimensional Boltzmann equation.** We perform a series of kinetic Monte Carlo simulations for the Boltzmann-type models presented in Sections 2.2.1 and 2.2.2. In this kind of simulation, known as direct simulation Monte Carlo (DSMC) or Bird's scheme, pairs of individuals are randomly and non-exclusively selected for binary collisions, and exchange opinion (and assertiveness in the multi-dimensional model from Section 2.2.2), according to the relevant interaction rule.

In each simulation we consider $N = 5000$ individuals, which are uniformly distributed in $\mathcal{I}\times\mathcal{J}$ at time $t = 0$. One time step in our simulation corresponds to N interactions. The average steady state opinion distribution $f = f(x, w, t)$ is calculated using $M = 10$ realisations. To compute a good approximation of the steady state, each realisation is carried out for $n = 2\times 10^6$ time steps, then the particle distribution is averaged over 5000 time steps. The random variables are chosen such that η_i and μ_i assumes only values $\nu = \pm 0.02$ with equal probability. We assume that the diffusion has the form $D(w) = (1 - w^2)^\alpha$ to ensure that the opinion w remains inside the interval $\mathcal{I}$. The parameter α is set to $\alpha = 2$ if not mentioned otherwise.

5.2. **Probabilistic Monte Carlo simulations for the inhomogeneous Boltzmann equation.** The Monte Carlo simulations for the inhomogeneous Boltzmann-type equations (2) are more involved. In equation (2) the advection term is of conservative form, hence the transportation step cannot be translated directly to the particle simulation. Pareschi and Seaid [21] as well as Herty *et al.*[18] propose a Monte Carlo method, that is based on a relaxation approximation of conservation laws. It corresponds to a semilinear system with linear characteristic variables. We shall briefly review the underlying idea for the conservative transportation operator in (2) in the following.

Let us consider the linear conservation law in one spatial dimension for the function $\rho = \rho(x,t)$ with a given flux function $b(x,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$:

$$\partial_t \rho + \partial_x (b(x,t)\rho) = 0, \;\; \rho(x,0) = \rho_0(x). \tag{29}$$

Note that (29) corresponds to the transportation step in a splitting scheme in a Monte Carlo simulation for (2). Equation (29) can be approximated by the following semilinear relaxation system:

$$\partial_t \rho + \partial_x v = 0, \quad \partial_t v + \bar{b}\partial_x u = -\frac{1}{\varepsilon}(v - b(x,t)\rho) \tag{30}$$

with initial conditions $\rho(x,0) = \rho_0(x)$ and $v(x,0) = b(x,0)\rho_0(x)$ and $\bar{b}, \varepsilon > 0$. The function v approaches the solution at local equilibrium $v = \varphi(u)$, if $-\bar{b} \le b(x,t) \le \bar{b}$ for all x. Note that system (30) has two characteristic variables given by $v \pm \sqrt{\bar{b}}$, which correspond to particles either moving to the left or the right with speed $\sqrt{\bar{b}}$. Hence, we introduce the kinetic variables p and q with

$$\rho = p + q \text{ and } v = \bar{b}(p - q).$$

Then the relaxation system can be written as follows:

$$\partial_t p + \partial_x\big(\sqrt{\bar{b}}p\big) = \frac{1}{\varepsilon}\Big(\frac{q-p}{2} + \frac{\varphi(p+q)}{2\sqrt{\bar{b}}}\Big), \quad \partial_t q - \partial_x\big(\sqrt{\bar{b}}q\big) = \frac{1}{\varepsilon}\Big(\frac{p-q}{2} - \frac{\varphi(p+q)}{2\sqrt{\bar{b}}}\Big). \tag{31}$$

The numerical solver is based on a splitting algorithm for system (31). It corresponds to first solving the transportation problem and then the relaxation step. While the transportation step is straight forward, the relaxation step can be interpreted as the evolution of a probability density. Since

$$p \ge 0, \quad q \ge 0, \quad \frac{p}{\rho} + \frac{q}{\rho} = 1 \text{ and } \partial_t \rho = 0$$

in the relaxation step we can calculate the solution explicitly and obtain

$$p = \frac{1}{2}e^{\frac{t}{\varepsilon}} + \frac{1}{2\bar{b}}b(x,t)\rho \quad \text{ and } \quad q = -\frac{1}{2}e^{\frac{t}{\varepsilon}} - \frac{1}{2\bar{b}}b(x,t).$$

Then the solution at the time $t^{n+1} = (n+1)\Delta t$ reads as:

$$p(x,t^{n+1}) = (1-\lambda)p(x,t^n) + \lambda b(x,t^n)\rho(x,t^n), \quad q(x,t^{n+1}) = (1-\lambda)q(x,t^n) - \lambda b(x,t^n)\rho(x,t^n),$$

with $\lambda = 1 - e^{-\Delta t/\varepsilon}$. Let $p^n = p(x,t^n)$, $q^n = q(x,t^n)$ and $\rho^n = p^n + q^n$. Since $0 \le \lambda \le 1$ we can define the probability density

$$P^n(\xi) = \begin{cases} p^n/\rho^n & \text{if } \xi = \sqrt{\bar{b}}, \\ q^n/\rho^n & \text{if } \xi = -\sqrt{\bar{b}}, \\ 0 & \text{elsewhere.} \end{cases}$$

Since $0 \le P^n(\xi) \le 1$ and $\sum_\xi P^n(\xi) = 1$, the relaxation step can be interpreted as the evolution of a probability function

$$P^{n+1}(\xi) = (1-\lambda)P^n(\xi) + \lambda E^n(\xi),$$

where $E^n(\xi) = b(x,t^n)\rho^n$ if $\xi = \sqrt{\bar{b}}$ and $E^n(\xi) = -b(x,t^n)\rho^n$ if $\xi = -\sqrt{\bar{b}}$. So in the probabilistic Monte Carlo method a particle either moves to the right or the left with maximum speed $\sqrt{\bar{b}}$ in the transportation step. Then the two groups are resampled according to their distribution with

respect to the equilibrium solution $b(x,t^n)\rho^n$ in the relaxation step. For further details we refer to [21].

5.3. **Fokker-Planck simulations.** The discretisation of the Fokker-Planck-type equation (24) is based on a time splitting algorithm. The splitting strategy allows us to consider the interactions in the opinion variable, i.e. the right hand side of equation (24), and the transport step in space separately.

Let Δt denote the size of each time step and $t^k = k\Delta t$. Then the splitting scheme consists of a transport step $S^1(g,\Delta t)$ for a small time interval Δt:

$$\frac{\partial g^*}{\partial t}(x,w,t) + \operatorname{div}_x\big(\phi(x,w)g^*(x,w,t)\big) = 0, \tag{32a}$$

$$g^*(x,w,0) = g_0^*(x,w), \tag{32b}$$

and an interaction step $S^2(g,\Delta t)$:

$$\frac{\partial g^\diamond}{\partial t}(x,w,t) = \frac{\partial}{\partial w}\Big(\frac{1}{\tau}\mathcal{K}(x,w,t)g^\diamond(x,w,t)\Big) + \frac{\lambda M(x)}{2\tau}\frac{\partial^2}{\partial w^2}\big(D^2(w)g^\diamond(x,w,t)\big), \tag{33a}$$

$$g^\diamond(x,w,0) = g^*(x,w,\Delta t). \tag{33b}$$

The approximate solution at time $t = t^{k+1}$ is given by

$$g^{k+1}(x,w) = S_2(g^{*,k+1},\Delta t/2)\circ S_1(g^{\diamond,k+\frac{1}{2}},\Delta t)\circ S_2(g^k,\Delta t/2),$$

where the superscript indizes denote the solution of $g^\diamond$ and g^* at the discrete time steps $t^k = k\Delta t$ and $t^{k+\frac{1}{2}} = (k+\frac{1}{2})\Delta t$. Both equations are solved using an explicit Euler scheme in time and a conforming finite element discretisation with linear basis functions $p_i, i = 1,\dots$ (#points), in space as well as opinion. Then the discrete transportation step (32) in space has the form

$$\mathbf{M}\Big[g_l^{k+1}(x,\cdot) - g_l^k(x,\cdot)\Big] = \Delta t\Big(\mathbf{T}\big[g_l^k(x,\cdot)\big]\Big),$$

where $\mathbf{M} = \langle p_i, p_j\rangle_{ij}$ corresponds to the mass matrix for element-wise linear basis functions and $\mathbf{T} = \langle \Phi(x)\nabla p_i, p_j\rangle_{ij}$ denotes the matrix corresponding to the convective field Φ of the form (9). In the interaction step we approximate the mass matrix by the corresponding lumped mass matrix $\tilde{\mathbf{M}}$, which can be inverted explicitly. The discrete interaction step (33) reads as:

$$\tilde{\mathbf{M}}\Big[g_l^{k+1}(\cdot,w) - g_l^k(\cdot,w)\Big] = \Delta t\Big(\frac{1}{\tau}\mathbf{C}\big[K[g_l^k], g_l^k(\cdot,w)\big] + \frac{\lambda M(\cdot)}{2\tau}\mathbf{L}\big[g_l^k(\cdot,w)\big]\Big),$$

with the discrete convolution-transportation matrix $\mathbf{C} = \langle K(x,w)p_i, \nabla p_j\rangle$ and Laplacian $\mathbf{L} = \langle D(w)\nabla p_i, \nabla p_j\rangle$. Here the vector K corresponds to the discrete convolution operator $\mathcal{K}$, which has been approximated by the midpoint rule:

$$K(x,w_l) = \sum_k C(x,v_k,w_l)(w_l - v_k)g(x,v_k)\Delta w.$$

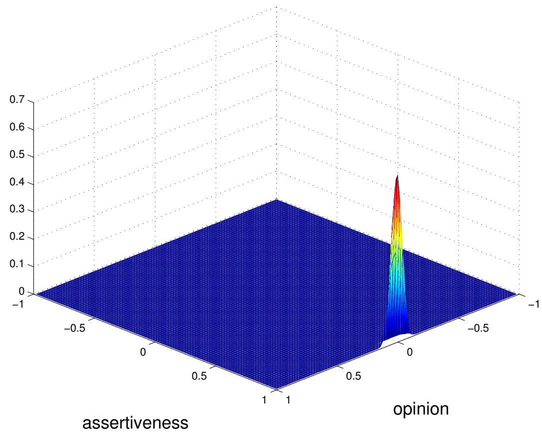
5.4. **Numerical results: opinion formation and opinion leadership.** We present numerical results for the kinetic models for opinion formation including the assertiveness of individuals from Section 2. We consider the inhomogeneous Boltzmann-type model of Section 2.2.1 and the multi-dimensional Boltzmann-type equation of Section 2.2.2 which are discretised using the Monte Carlo methods presented in the previous sections.

5.4.1. *Example 1: Influence of the interaction radius r.* First we would like to illustrate the influence of the interaction radius r. We assume that the functions in both models which relate to the increase or decrease of the assertiveness (see (9) and (12)) define a constant increase of the individual assertiveness level, i.e.

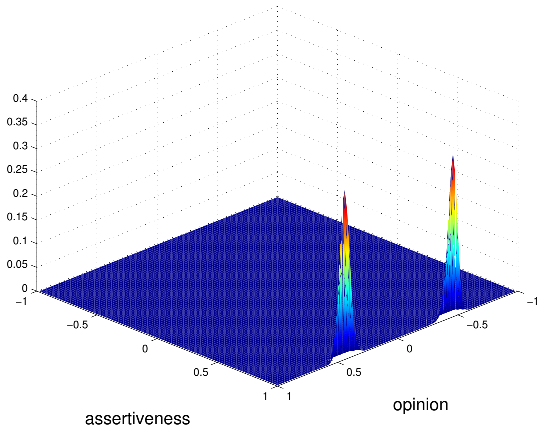
$$G(x,w) = 1 \text{ and } \tilde{G}(x,x-y) = (1-x^2)^\alpha \tanh\big(k(x-y)\big),$$

with $k = 10$. We start each Monte Carlo simulation with an equally distributed number of individuals within $(v,x) \in (-1,1)\times(-1,1)$. We expect the formation of one or several peaks at the highest assertiveness level due to the constant increase caused by the functions G and $\tilde{G}$. Figure 2, which shows the averaged steady state densities for $\delta = \gamma = 0.25$ and two different radii,

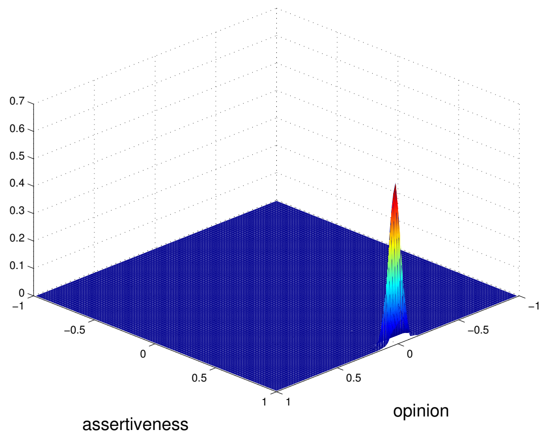
$r = 0.5$ and $r = 1$. For $r = 1$ we observe the formation of a single peak located at the highest assertiveness level, $w = 1$, in Figure 2 in the inhomogeneous and multi-dimensional model. In the case of the smaller interaction radius, $r = 0.5$, two peaks form at the highest assertiveness level. These results are in accordance with numerical simulations of the original Toscani model (3).



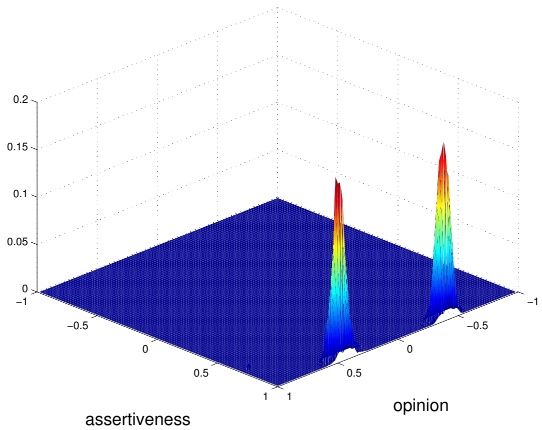
(a) Inhomog. $r = 1$



(b) Inhomog. $r = 0.5$



(c) Multi-dim. $r = 1$



(d) Multi-dim $r = 0.5$

FIGURE 2. Example 1: Steady state densities of the inhomogeneous and multidimensional Boltzmann-type opinion formation model for different interaction radii r.

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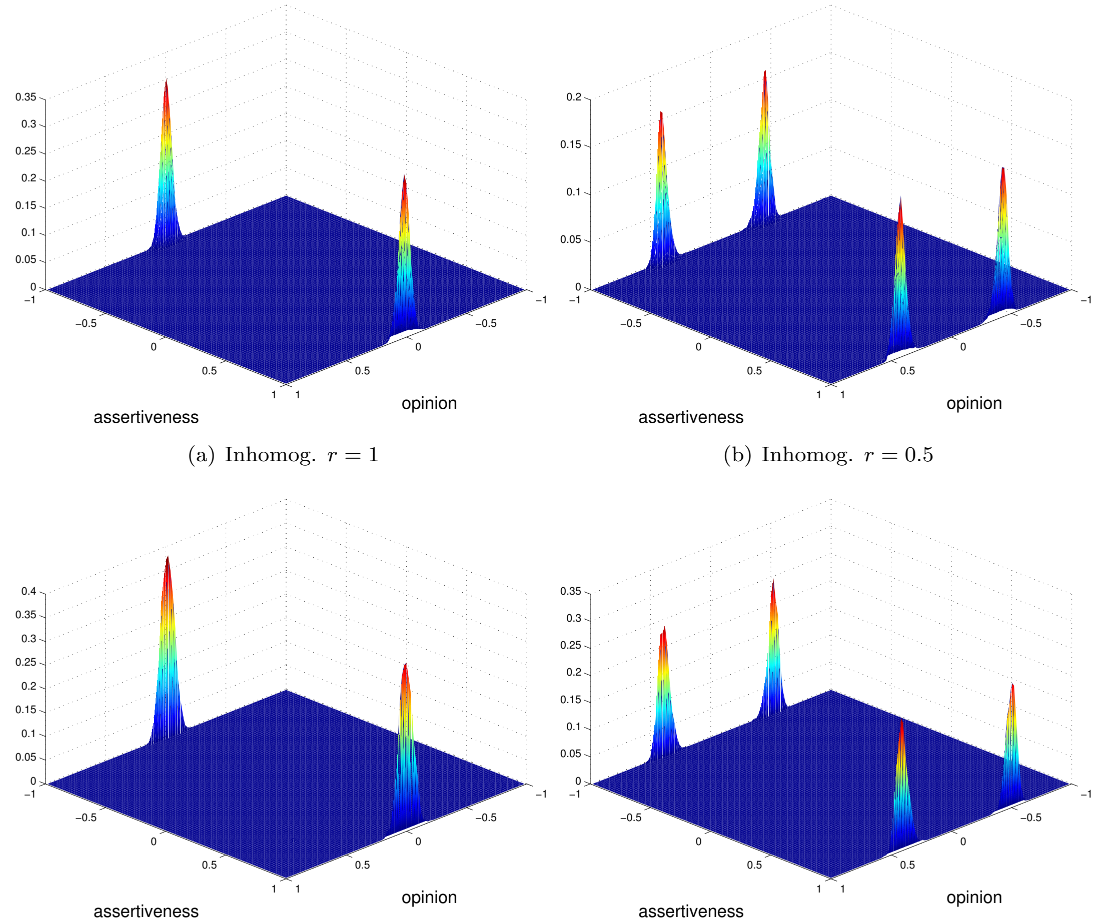
5.4.2. *Example 2: Different choices of G and $\tilde{G}$.* Next we focus on the influence of the functions G and $\tilde{G}$, which model the increase of the assertiveness level in either model (see (9) and (12)). We choose the same initial distribution as in Example 1 and set

$$G(x, w) = \tanh(kx) \text{ and } \tilde{G}(x, x - y) = (1 - x^2)^\alpha |x - y|.$$

Both functions are based on the following assumption: individuals with a high assertiveness level gain confidence, while those with a lower level loose. We expect the formation of peaks close to the highest and lowest assertiveness level. This assumption is confirmed by our numerical experiments, see Figure 3. We observe the formation of peaks close to the maximum and minimum assertiveness level; the number of peaks again depends on the interaction radius r.



(a) Inhomog. $r = 1$

(b) Inhomog. $r = 0.5$

(c) Multi-dim. $r = 1$

(d) Multi-dim $r = 0.5$

Figure 3. Example 2: Steady state densities of the inhomogeneous and multidimensional Boltzmann-type opinion formation model for different interaction radii.

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5.4.3. *Example 3: Preferred opinions promoting leadership.* Our last example illustrates the rich behaviour of the proposed models. We assume that leadership qualities are directly related to the opinion of an individual. Individuals with a popular, 'mainstream' opinion, i.e. $w = \pm 0.5$ gain confidence, while extreme opinions are not promoting leadership. To model this we set

$$G(x,w) = \tilde{G}(x,w,x-y) = \frac{1}{\sqrt{2\pi}\sigma}\left(e^{-\frac{1}{2\sigma^2}(w-0.5)^2} + e^{-\frac{1}{2\sigma^2}(w+0.5)^2}\right)$$

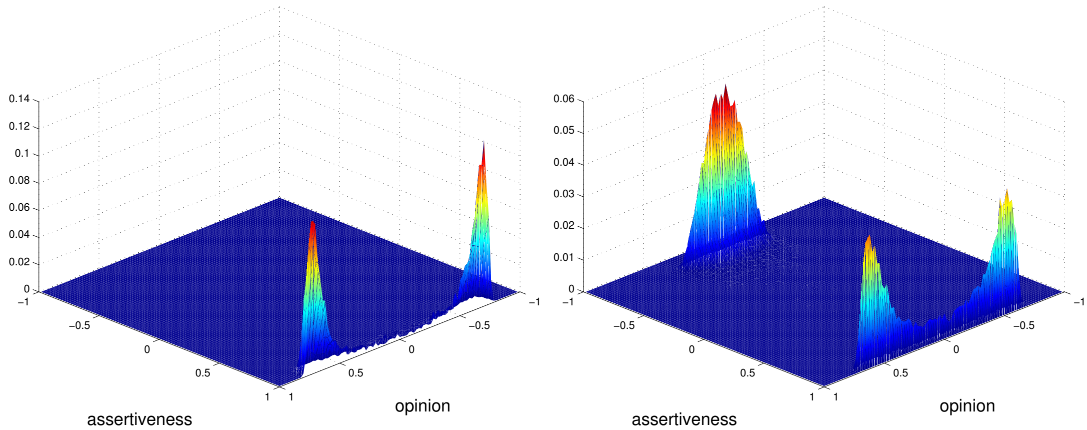
with $\sigma^2 = 1/8$ and

$$R(x,x-y) = \left(\frac{1}{2} - \frac{1}{2}\tanh\big(k(x-y)\big)\right)e^{-(x+1)^2} \tag{34}$$

with $k = 10$. The second factor in (34) models the assumption that individuals change their opinion more if they have a low assertiveness level.

At time $t = 0$ we equally distribute the individuals within $(w,x) \in [-0.25, 0.25] \times [-0.75, -0.25]$ – hence we assume that initially no extreme opinions exist and no leaders are present. The interaction radius is set to $r = 1$. Figure 4 illustrates the very interesting behaviour in this case. In the multidimensional simulation we observe the formation of a single peak at a low assertiveness

level, while individuals with a higher assertiveness level group around the ‘mainstream’ opinion $w = \pm 0.5$. In the case of the inhomogeneous model, the potential Φ initiates an increase of the assertiveness level for all individuals. Therefore we do not observe the formation of a single peak at a low assertiveness level in this case.



(a) Inhomog. (b) Multi-dim.

FIGURE 4. Example 3: Steady state densities of the inhomogeneous and multi-dimensional Boltzmann-type opinion formation model.

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5.5. **Numerical results: ‘The Big Sort’ in Arizona.** In our final example we present numerical simulations of the corresponding Fokker-Planck equation (24), which illustrate ‘The Big Sort’ by considering the state of Arizona. Arizona is a state in the southwest of the United States with 15 electoral counties. In recent years the Republican Party dominated Arizonas politics, see for example the outcome of the presidential elections from the years 1992 to 2004 in Figure 5. The colours red and blue correspond to Republicans and Democrats, respectively. The colour intensity reflects the election outcome in percent, i.e. dark blue corresponds to Democrats 60–70%, medium blue to Democrats 50–60% and light blue to Democrats 40–50%. Similar colour codes are used for the Republicans. The election results illustrate the clustering trend as the election results per county become more and more pronounced over the years.

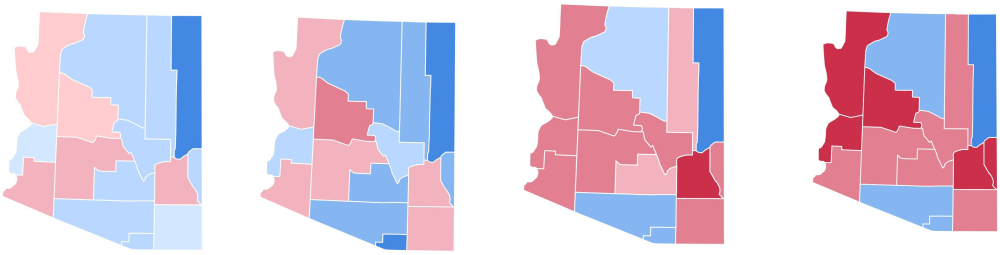
We solve the Fokker-Planck equation (24) on a bounded physical domain $\Omega \subset \mathbb{R}^2$ corresponding to the state Arizona, divided into 15 electoral counties, see Figure 6(a). We employ the numerical strategy outlined in Section 5.5.3. We discretise the physical domain in 9356 triangles, the opinion domain $\mathcal{J} = [-1, 1]$ into 100 elements. The time steps are set to $\Delta t = 2 \times 10^{-4}$.
We choose an initial distribution which is proportional to the election result in 1992,

$$f(x, w, 0) = 0.5 + \begin{cases} w(1-w) f_{D,1992}(x) & \text{for } w > 0, \\ w(1+w) f_{R,1992}(x) & \text{for } w < 0, \end{cases}$$

where $f_{D,1992}$ and $f_{R,1992}$ correspond to the distribution of Democrats and Republicans estimated from the election results in 1992. We approximate the distributions by assigning different constants to the respective percentages in the electoral vote, i.e.

$$f_{D,R}(x) = \begin{cases} 0.25 & \text{if the election outcome is } < 40 - 50\% \\ 0.5 & \text{if the election outcome is between } 50 - 60\% \\ 0.75 & \text{if the election result is } > 60\%. \end{cases} \tag{35}$$

The potential Φ, given by (15), attracts individuals to the counties controlled by the party they support. We assume that the potential $C = C(x)$ is directly related to the electoral results of the



(a) 1992 (b) 1996 (c) 2000 (d) 2004

FIGURE 5. Results of the presidential elections in Arizona [19]. The colour intensities reflect the election outcome in percent, i.e. dark blue (red) to Democrats (Republicans) 60–70%, medium blue (red) to Democrats (Republicans) 50–60% and light blue (red) to Democrats (Republicans) 40–50%.

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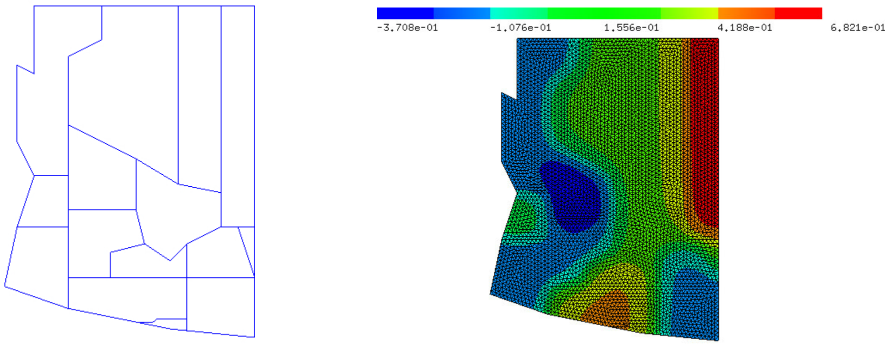


(a) Computational domain Ω (b) Potential C.

FIGURE 6. Computational domain Ω representing the 15 electoral counties of Arizona and potential C corresponding to the election results in 1996.

year 1996, and satisfies the following PDE:

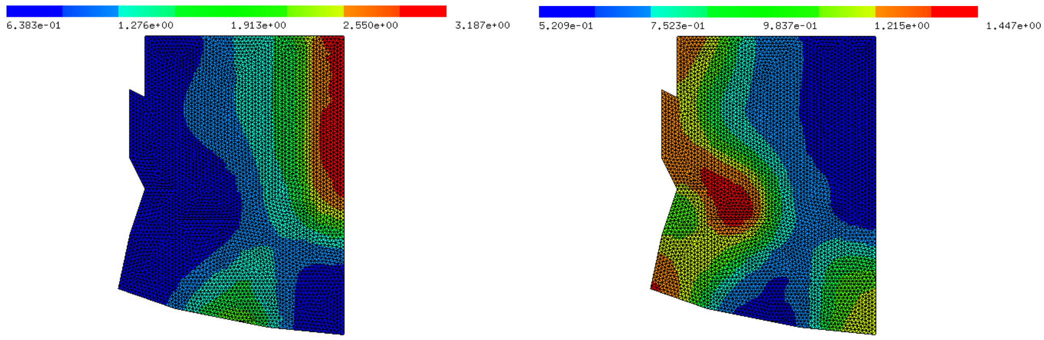
$$C(x) + \varepsilon \Delta C(x) = f_{1996}(x) \quad \text{for all } x \in \Omega, \quad \nabla C(x) \cdot n = 0 \quad \text{for all } x \in \partial\Omega.$$

The Laplacian is added to smooth the potential C, the right hand side corresponds to the election results in the year 1996 (using the same constants as in (35) with $\pm$ sign corresponding to the Republicans and Democrats respectively). Note that we choose Neumann boundary conditions to ensure that individuals stay inside the physical domain. The calculated potential C is depicted in Figure 6(b). The simulation parameters are set to $\lambda = 0.1$, $\tau = 0.25$ and the interaction radius to $r = 5$.

Figure 7 shows the spatial distribution of the marginals (17), i.e.

$$f_D(x,t) = \int_{-1}^{0} f(x,w,t)\,dw \quad \text{and} \quad f_R(x,t) = \int_{0}^{1} f(x,w,t)\,dx,$$

which correspond to Democrats and Republican, respectively, at time $t = 0.5$. We observe the formation of two larger clusters of democratic voters in the south and the Northeast of Arizona. The republicans move towards the Northwest as well as the Southeast. This simulation results reproduce the patterns of the electoral results from 2004 fairly well, except for the second county from the right in the Northeast (Navajo). However, given the electoral data from 1992 and 1996 only it is not possible to initiate such opinion dynamics. This would require more in-depth



(a) Distribution of democrats

(b) Distribution of republicans

FIGURE 7. The Big Sort: distribution of Democrats, $f_D(x,t) = \int_{-1}^{0} f(x,w,t)\,dw$, and Republicans, $f_R(x,t) = \int_0^1 f(x,w,t)\,dw$.

information such as the demographic distribution within the counties or the incorporation of several sets of electoral results. We leave such extensions for future research.

6. Conclusion

We proposed and examined different inhomogeneous kinetic models for opinion formation, when the opinion formation process depends on an additional independent variable. Examples included opinion dynamics under the effect of opinion leadership and opinion dynamics modelling political segregation. Starting from microscopic opinion consensus dynamics we derived Boltzmann-type equations for the opinion distribution. In a quasi-invariant opinion limit they can be approximated by macroscopic Fokker-Planck-type equations. We presented numerical experiments to illustrate the models' rich behaviour. Using presidential election results in the state of Arizona, we showed an example modelling the process of political segregation in the 'The Big Sort', the process of clustering of individuals who share similar political opinions.

Acknowledgements

MTW acknowledges financial support from the Austrian Academy of Sciences (ÖAW) via the New Frontiers Grant NFG0001.
The authors are grateful to Prof. Richard Tsai (UT Austin) for suggesting 'The Big Sort' as an example for an inhomogeneous opinion formation process.

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[1] Department of Mathematics, University of Sussex, Brighton, BN1 9QH, United Kingdom, Email: b.during@sussex.ac.uk, [2] Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, 4040 Linz, Austria, Email: mt.wolfram@ricam.oeaw.ac.at